Unit - III

Finite Fourier transforms:

If $f(x)$ is defined in the interval $(0, l)$ then finite fourier sine transform of $f(x)$ is given by

$$\boxed{F_s[f(x)] = \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx}$$

Inverse finite fourier sine transform of $F_s^{-1}[f(x)]$ given by.

$$\boxed{f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s[f(x)] \sin \frac{n\pi x}{l}}$$

If $f(x)$ is defined in the interval $0 < x < l$, than finite fourier cosine transform is defined by

$$\boxed{F_c\left[f(x)\right] = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx}$$

Inverse finite fourier cosine transform is defined by

$$\boxed{f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_c\left[f(x)\right]\frac{\cos n\pi x}{l} + \frac{1}{l}F_c(0)}$$

Problem 1:

Find fourier sine and cosine transform for the function $f(x) = x^2$ in $0 < x < l$

Solution:

Finite fourier sine transform is

$$F_s\left[f(x)\right] = \int_0^l f(x)\,\frac{\sin n\pi x}{l}\cdot dx$$

$$F_s\left[x^2\right] = \int_0^l x^2 \sin\frac{n\pi x}{l}\,dx$$

u, v_1, u', v_2

$$= \left[x^2\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 2x\left(\frac{-\frac{\sin n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + \frac{2\cos\frac{n\pi x}{l}}{\left(n\pi/l\right)^3}\right]_0^l$$

$$= \left[\frac{\frac{-x^2\cos n\pi x}{l}}{n\pi/l} + \frac{\frac{2x\sin n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} + \frac{\frac{2\cos n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \left[\frac{-l^3\cos n\pi}{n\pi} + \frac{2l^3\sin n\pi}{n^2\pi^2} + \frac{2l^3\cos n\pi}{n^3\pi^3}\right] - \left(0 + \frac{2l^3}{n^3\pi^3}\right)$$

$\sin n\pi = 0$, $\cos 0 =$

$$= -\frac{l^3}{n\pi}(-1)^n + \frac{2l^3}{n^3\pi^3}(-1)^n - \frac{2l^3}{n^3\pi^3}$$

then w.r.t

$$F_c\left[f(x)\right] = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

$\int u\,dv = uv_1 - u'v_2 + u''v_3$

$u = x^2$, $u' = 2x$, $u'' = 2$, $u''' = 0$

$v_1 = \frac{\sin n\pi x / l}{n\pi / l}$, $v_2 = \frac{-\cos n\pi x / l}{n^2\pi^2/l^2}$, $v_3 = \frac{-\sin n\pi x / l}{n^3\pi^3/l^3}$

(ie) $$F_c\left[x^2\right] = \left[\frac{x^2\left(\sin\frac{n\pi x}{l}\right)}{n\pi/l} - 2x\left(\frac{-\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) + 2\left(\frac{-\sin\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= \left[\frac{l^3\sin n\pi}{n\pi} + \frac{2l^3\cos n\pi}{n^2\pi^2} - \frac{2l^3\sin n\pi}{n^3\pi^3}\right]$$

$$= \frac{2l^3}{n^2\pi^2}\left[(-1)^n\right]$$

$$\Rightarrow F_c\left[x^2\right] = \frac{2l^3}{n^2\pi^2}(-1)^n$$

(2) Find finite fourier sine and cosine transform of $f(x) = x$ in the interval $(0,\pi)$

Solution:.

$$F_s\left[f(x)\right] = \int_0^l f(x)\,\frac{\sin n\pi x}{l}\,dx$$

$$= \int_0^\pi x\,\frac{\sin n\cancel{\pi} x}{\cancel{\pi}}\,dx \qquad l = \pi$$

$\int u\,dv = uv_1 - u'v_2 + u''v_3$

$u = x$, $dv = \sin nx\,dx$

$v_1 = \frac{-\cos nx}{n}$, $v_2 = \frac{-\sin nx}{n^2}$

$$= \int_0^\pi x\sin nx\,dx$$

$$= \left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^\pi$$

$$= \pi\left(\frac{-\cos n\pi}{n}\right) = \frac{-\pi}{n}(-1)^n$$

$$(-1)^n(-1)\frac{\pi}{n}$$

$$(-1)^{n+1}$$

$$= (-1)^{n+1}\,\pi/n$$

$$F_s[x] = (-1)^{n+1}\,\pi/n$$

then we.k.t

$$F_c[f(x)] = \int_0^l f(x)\,\frac{\cos n\pi x}{l}\,dx$$

(i.e) $F_c[x] = \int_0^{\pi} x\,\frac{\cos n\pi x}{\pi}\,dx$ $\quad [l=\pi]$

$$= \int_0^{\pi} x\cos nx\,dx$$

$$= \left[x\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] = \frac{(-1)^n}{n^2} - \frac{1}{n^2}$$

$$= \frac{1}{n^2}\left[(-1)^n - 1\right]$$

$$F_c[x] = \frac{1}{n^2}\left[(-1)^n - 1\right]$$

③ Find finite fourier sine and cosine transform for $f(x) = e^{ax}$ in $(0, l)$

Solution:

$$F_s[f(x)] = \int_0^l f(x)\,\sin\frac{n\pi x}{l}\,dx \qquad b = \frac{n\pi x}{l} \quad a = a$$

$$F_s[e^{ax}] = \int_0^l e^{ax}\sin\frac{n\pi x}{l}\,dx$$

$$= \left[\frac{e^{al}}{\frac{al^2 + n^2\pi^2}{l^2}} \left\{ a \sin n\pi l - \frac{n\pi}{l} \cdot \cos \frac{n\pi l}{l} \right\} \right.$$

$$\left. - \frac{e^0}{\frac{l^2 a^2 + n^2\pi^2}{l^2}} \left\{ a \sin 0 - \frac{n\pi}{l} \cos 0 \right\} \right]$$

$$= \frac{l^2 e^{al}}{a^2 l^2 + n^2\pi^2} \left\{ \frac{-n\pi}{l} \cos n\pi \right\} - \frac{l^2}{l^2 a^2 + n^2\pi^2} \left[\frac{n\pi}{l} \right]$$

$$= \frac{l^2}{l^2 a^2 + n^2\pi^2} \left[(-1)^{n+1} e^{al} \frac{n\pi}{l} + \frac{n\pi}{l} \right]$$

$$= \frac{l^2}{l^2 a^2 + n^2\pi^2} \left(\frac{n\pi}{l} \right) \left[(-1)^{n+1} e^{al} + 1 \right]$$

$$F_s\left[e^{ax}\right] = \frac{l n\pi}{l^2 a^2 + n^2\pi^2} \left[(-1)^{n+1} e^{al} + 1 \right]$$

then w.k.t

$$F_c\left[f(x)\right] = \int_0^l f(x) \frac{\cos n\pi x}{l} \, dx$$

(i.e) $F_c\left[e^{ax}\right] = \int_0^l e^{ax} \frac{\cos n\pi x}{l} \, dx$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

$$= \left[\frac{e^{ax}}{\frac{a^2 l^2 + n^2\pi^2}{l^2}} \left[a \frac{\cos n\pi x}{l} + \frac{n\pi x}{l} \frac{\sin n\pi x}{l} \right] \right]_0^l$$

$$= \frac{e^{al}}{\frac{a^2 l^2 + n^2\pi^2}{l^2}} \left[\left\{ a \cos \frac{n\pi l}{l} + \frac{n\pi}{l} \sin n\pi \right\} - \frac{e^0}{\frac{a^2 l^2 + n^2\pi^2}{l^2}} \left\{ a \cos 0 + 0 \right\} \right]$$

$$= \left[\frac{l^2 e^{al}}{a^2 l^2 + n^2\pi^2} \left\{ a (-1)^n \right\} - \frac{l^2}{a^2 l^2 + n^2\pi^2} \left\{ a \right\} \right]$$

$$= \frac{1}{2}\left[\frac{n(-1)^n-(-1)^n+n(-1)^n+(-1)^n+2n}{n^2-1}\right]$$

$$= \frac{1}{2}\left[\frac{2n(-1)^n+2n}{n^2-1}\right]$$

$$= \frac{2n((-1)^n+1)}{2(n^2-1)} = \frac{n\left[(-1)^n+1\right]}{n^2-1}$$

Then we know that

$$F_c\left[f(x)\right] = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$F_c\left[\cos x\right] = \int_0^\pi \cos x\,\frac{\cos n\pi x}{\pi}\,dx$$

$\cos A\cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$

$$= \int_0^\pi \cos \overset{A}{nx}\,\cos \overset{B}{x}\,dx$$

$$= \frac{1}{2}\int_0^\pi \cos(nx+x)+(\cos nx-x)\,dx$$

$$F_c\left[\cos x\right] = \frac{1}{2}\int_0^\pi \cos(n+1)x+\cos(n-1)x\,dx$$

$$F_c\left[\cos x\right] = \frac{1}{2}\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_0^\pi$$

$$= \frac{1}{2}\left[\frac{\sin n\pi}{n+1}+\frac{\sin n\pi}{n-1}-\frac{\sin 0}{n+1}-\frac{\sin 0}{n-1}\right]$$

$$= 0$$

$$F_c\left[\cos x\right] = 0$$

Find $f(x)$ if $F_c(P) = \dfrac{6\sin\frac{P\pi}{2} - \cos P\pi}{(2P+1)\pi}$ for

$P = 1, 2 \ldots \ldots = 2/\pi$ for $P-0$, where $0<x<4$

soln

Inverse finite fourier cosine transform is

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c[f(x)] \cos\frac{n\pi x}{l}$$

Here, $l=4$, $n=P$ $F_c(0) = 2/\pi$

$$f(x) = 1/4\,(2/\pi) + 2/4 \sum_{P=1}^{\infty} F_c(P) \cos\frac{P\pi x}{4}$$

$$= \frac{1}{2\pi} + \frac{1}{2} \sum_{P=1}^{\infty} \frac{6\sin P\pi/2 - \cos P\pi}{(2P+1)\pi} \cdot \frac{\cos P\pi x}{4}$$

$$f(x) = + \frac{1}{2\pi}\left[1 + \sum_{P-1}^{\infty} \frac{6\sin P\pi/2 - \cos P\pi}{2P+1} \cdot \frac{\cos P\pi x}{4}\right]$$

Find finite fourier sine transform of $f(x) = \cos kx$ in $0<x<\pi$

soln:

We know that,

$$F_s[f(x)] = \int_0^l f(x) \frac{\sin n\pi x}{l} dx \quad \cancel{\frac{\sin n\pi x}{l}}$$

(i.e) $$F_s[\cos kx] = \int_0^\pi \cos kx \cdot \sin nx \, dx \quad (\because l=\pi)$$

$$= \int_0^\pi \cos kx \cdot \sin nx \, dx$$

$$= 1/2 \int_0^\pi \sin(nx+kx) + \sin(nx-kx)\, dx$$

$$= 1/2 \int_0^\pi \sin(n+k)x + \sin(n-k)x\, dx$$

$$= 1/2 \left[\frac{-\cos(n+k)x}{n+k} - \frac{\cos(n-k)x}{n-k}\right]_0^\pi$$

$$= \frac{1}{2}\left[\frac{\cos(n+k)\pi}{n+k} - \frac{\cos(n-k)\pi}{n-k} + \frac{1}{n+k} + \frac{1}{n-k}\right]$$

$$= \frac{1}{2}\left[-\frac{(\cos n\pi \cos k\pi - \sin n\pi \sin nk\pi)}{n+k} - \frac{(\cos n\pi \cos k\pi + \sin n\pi \sin k\pi)}{n-k} + \frac{1}{n+k} + \frac{1}{n-k}\right]$$

$$= \frac{1}{2}\left[\frac{-(-1)^n \cos k\pi}{n+k} - \frac{(-1)^n \cos k\pi}{n-k} + \frac{2^n}{n^2-k^2}\right]$$

$$= \frac{1}{2}\left[-(-1)^n \cos k\pi \left(\frac{1}{n+k} + \frac{1}{n-k}\right) + \frac{2n}{n^2-k^2}\right]$$

$$= \frac{1}{2}\left[-(-1)^n \cos k\pi \cdot \frac{2n}{n^2-k^2} + \frac{2n}{n^2-k^2}\right]$$

$$= \frac{2n}{2(n^2-k^2)}\left[1-(-1)^n \cos k\pi\right]$$

$$F_s[\cos kx] = \frac{n}{n^2-k^2}\left[1-(-1)^n \cos k\pi\right]$$

(7) Find $f(x)$ if its finite fourier sine tranform given by $F_s(P) = 1 - \frac{\cos p\pi}{p^2\pi^2}$, where $0<x<\pi$

$P = 1, 2 \ldots,$

Soln:

Inverse finite fourier sine transform is given by.

$$F(x) = 2/l \sum_{n=1}^{\infty} F_s[f(x)] \sin \frac{n\pi x}{l}$$

Here $l = \pi$, $n = P$

$$F(x) = 2/\pi^3 \sum_{P=1}^{\infty} \frac{1-\cos P\pi}{P^2} \sin px$$

$$\left[\because F(x) = 2/\pi \sum_{P=1}^{\infty} \frac{1-\cos p\pi}{p^2\pi^2} \frac{\sin p\pi x}{\pi}\right.$$

$$\left.\text{(i.e) } f(x) = 2/\pi^3 \sum_{P=1}^{\infty} \frac{1-\cos p\pi}{p^2} \sin px\right]$$

Q.2 Find $f(x)$ if its cosine transform is

$$F_c[s] = \begin{cases} \frac{1}{\sqrt{2\pi}}(a - s/2) & \text{if } s < 2a \\ 0 & \text{if } s > 2a \end{cases}$$

soln: Inverse fourier cosine transform is

$$f(x) = \sqrt{2/\pi} \int_0^{\infty} F_c[f(x)] \cos sx \, ds$$

$$= \sqrt{2/\pi} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} (a - s/2) \cos sx \, dx$$

$$= 1/\pi \int_0^{2a} (a - s/2) \cos sx \, ds + \int_{2a}^{\alpha} (0)$$

$\int u\,dv = uv_1 - u'v_2$

$$= 1/\pi \left[(a - s/2)\left(\frac{\sin sx}{x}\right) - (-1/2)\left(\frac{-\cos sx}{x^2}\right) \right]_0^{2a}$$

$$= 1/\pi \left[-1/2 \frac{\cos 2ax}{x^2} + 1/2 \cdot 1/x^2 \right]$$

$$= 1/\pi \left[(1/2x^2)(1 - \cos 2ax) \right]$$

$$= \frac{1}{2\pi x^2} \left[2\sin^2\left(\frac{2ax}{2}\right) \right] \quad \left[\because 1 - \cos 2xa = 2\sin^2\left(\frac{2xa}{2}\right) \right]$$

$$= \frac{1}{2\pi x^2} \left[2\sin^2 \frac{2ax}{2} \right]$$

$$= \frac{1}{\pi x^2} (\sin^2 ax)$$

$$F(x) = \frac{1}{\pi x^2} (\sin^2 ax)$$

Q. Find fourier cosine transform of $f(x) = \pi/3 - x + x^2/2\pi$ in $(0, \pi)$

$V = \cos nx$

$V_1 = \frac{\sin nx}{n}$ $\quad V_2 = -\frac{\cos nx}{n^2}$

soln:

We know that,

$$F_c[f(x)] = \int^{l} f(x) \frac{\cos n\pi x}{l} dx$$

$$= \left[\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-1 + \frac{x}{\pi}\right)\left(-\frac{\cos n\pi}{n^2}\right) + \left(\frac{1}{\pi}\right)\left(\frac{-\sin nx}{n^3}\right) \right]_0^{\pi}$$

$$= \left\{ \left[\left(\frac{\pi}{3} - \pi + \frac{\pi^2}{2\pi}\right)(0) - 0 + 0 \right] - \left[1\left(\frac{-1}{n^2}\right) \right] \right\}$$

$$= \left\{ \frac{1}{n^2} \right\}$$

$$\therefore F_c\left[f(x)\right] = \frac{1}{n^2} \qquad n = 1, 2 \ldots$$

(10). Find the finite fourier cosine transform of $f(x) = \cos ax$ in $(0, l)$

soln:

Finite fourier cosine transform of $f(x)$ is

$$F_c\left[f(x)\right] = \int_0^l f(x) \frac{\cos n\pi x}{l}\, dx$$

$$= \int_0^l \cos ax \frac{\cos n\pi x}{l}\, dx$$

$$= \int_0^l \frac{1}{2}\left[\cos\left(\frac{ax + n\pi x}{l}\right) + \cos\left(ax - \frac{n\pi}{l}\right)\right] dx$$

$$= \frac{1}{2}\int_0^l \left\{\cos\left(a + \frac{n\pi}{l}\right)x + \cos\left(a - \frac{n\pi}{l}\right)x\right\} dx$$

$$= \frac{1}{2}\left[\frac{\sin\left(a + \frac{n\pi}{l}\right)x}{a + \frac{n\pi}{l}} + \frac{\sin\left(a - \frac{n\pi}{l}\right)x}{a - n\pi/l}\right]_0^l$$

$$= \frac{1}{2}\left[\frac{\sin(al + n\pi)}{a + n\pi/l} + \frac{\sin(al - n\pi)}{a - n\pi/l}\right]$$

$$= \frac{1}{2}\left[\frac{\sin al \cos n\pi + \cos al \sin n\pi}{a + n\pi/l} + \frac{\sin al \cos n\pi - \cos al \sin n\pi}{a - n\pi/l}\right]$$

$$= \frac{1}{2}\left[\frac{\sin al \cos n\pi + \cos al \sin n\pi + \sin al \cos n\pi - \cos al \sin n\pi}{(a+n\pi/l)(a-n\pi/l)}\right]$$

$$= \frac{1}{2}\left[\frac{\sin al\,(-1)^n + \sin al\,(-1)^n}{a^2 - n^2\pi^2/l^2}\right]$$

$$= \frac{1}{2}\left[\frac{2\sin al\,(-1)^n}{\frac{l^2a^2 - n^2\pi^2}{l^2}}\right]$$

$$= \frac{(-1)^n a^2 l^2 \sin al}{a^2l^2 - n^2\pi^2}$$

11. Find Fourier sine and cosine Tran

$f(x) = x \quad 0 < x < l.$

Finite fourier sine transform of $f(x)$.

$$F_s[f(x)] = \int_0^l f(x) \cdot \frac{\sin n\pi x}{l}\, dx$$

$$= \int_0^l x \cdot \frac{\sin n . lx}{l}\, dx$$

$$= \int_0^l x \sin nx\, dx$$

$u = x$

$v = \int \sin nx\, dx$

$v_1 = -\frac{\cos nx}{n}$

$u' = 1$

$$F_s\left[f(x)\right] = \left\{ \frac{x\left(-\frac{\cos n\pi x}{l}\right)}{n\pi/l} + \frac{\frac{\sin n\pi x}{l}}{n^2\pi^2/l^2} \right]_0^l$$

$$F_s\left[f(x)\right] = \left[\frac{-l\cos n\pi}{n\pi/l} + \frac{\sin n\pi}{n^2\pi^2/l^2} \right]$$

$$= \left[-\frac{\cos n\pi . l^2}{n\pi} + \frac{l^2 \sin n\pi}{n^2\pi^2} \right]$$

$$\left[\sin n\pi = 0\right]$$

$$F_s\left[f(x)\right] = \frac{-l^2(-1)^n}{n\pi}$$

$$= \frac{(-1)^{n+1} l^2}{n\pi}$$

$$\therefore F_s\left[f(x)\right] = \frac{l^2}{n\pi}(-1)^{n+1}$$

then we know that,

$$F_c\left[f(x)\right] = \int_0^l f(x) \frac{\cos n\pi x}{l} dx.$$

i.e) $$F_c\left[f(x)\right] = \int_0^l x \cdot \frac{\cos n\pi x}{l} dx$$

$u = x$ | $v = \int \frac{\cos n\pi x}{l}$

$u' = 1$ | $v_1 = +\frac{\sin n\pi x}{l} \Big/ n\pi/l$

$u'' = 0$ | $v_2 = -\frac{\frac{\cos n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$

$$F_c\left[f(x)\right] = \left[\frac{x\left(\frac{\sin n\pi x}{l}\right)}{n\pi} + \frac{\cos n\pi x/l}{n^2\pi^2} \right]^l$$

$$= \left[0 + \frac{\cos n\pi}{n^2\pi^2}(l^2) - \frac{1}{n^2\pi^2/l^2} \right]$$

$$= \left[\frac{(-1)^n \cdot l^2}{n^2\pi^2} - \frac{l^2}{n^2\pi^2} \right]$$

$$F_c\left[f(x)\right] = \frac{l^2}{n^2\pi^2}\left[(-1)^n - 1\right].$$

cosine transfrom.

Find Fourier transform of e^{iat} $(f(t))$ terms of fourier transforms of $F(t)$.

Solution:

Fourier transform of $F(t)$ is

$$F(S) = F[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f\, t\, e^{ist}\, dt$$

$$F[e^{iat} f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{iat} f(t) e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(t)\, e^{i(s+a)}\, dt$$

$$= F(s+a).$$

Find finite fourier cosine transform for $f(x) = (1 - x/\pi)^2$ in $0 < x < \pi$.

Solution:

Finite fourier cosine transform

$$F_c[f(x)] = \int_0^l f(x) \frac{\cos n\pi x}{l} dx$$

Here $l = \pi$.

$$\text{i.e) } F_c[f(x)] = \int_0^{\pi} f(x) \frac{\cos n \cancel{l} x}{\cancel{l}} dx$$

$$F_c[f(x)] = \int_0^{\pi} f(x) \cos nx\, dx.$$

$$= \int_0^{\pi} (1 - x/\pi)^2 \cos nx\, dx$$

$$(a-b)^2 = a^2 + b^2 - 2ab$$

$$= \int^{\pi} \left(1 + \frac{x^2}{\pi^2} - \frac{2x}{\pi}\right) \cos nx\, dx$$

let $u = 1 + \frac{x^2}{\pi^2} - \frac{2x}{\pi}$ | $v = \int \cos nx\,dx$

$u' = \frac{2x}{\pi^2} - \frac{2}{\pi}$ | $v_1 = \frac{\sin nx}{n}$

$u'' = \frac{2}{\pi^2}$ | $v_2 = +\frac{\cos nx}{n^2}$

$v_3 = -\frac{\sin nx}{n^3}$

$$\Rightarrow \int u\,dv = uv_1 - u'v_2 + u''v_3$$

$$= \left[\left(1 + \frac{x^2}{\pi^2} - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{2x}{\pi^2} - \frac{2}{\pi}\right)\left(\frac{\cos nx}{n^2}\right) + \left(\frac{2}{\pi^2}\right)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \left\{\left[\left(1 + \frac{\pi^2}{\pi^2} - \frac{2\pi}{\pi}\right)\left(\frac{\sin \pi}{n}\right) - \left(\frac{2\pi}{\pi^2} - \frac{2}{\pi}\right)\left(\frac{\cos n\pi}{n^2}\right) - \left(\frac{2}{\pi^2}\right)\left(-\frac{\sin n\pi}{n^3}\right)\right] - \left(0 - 0 - \frac{2\sin 0}{\pi^2 \cdot n^3}\right)\right]\right\}$$

$$= \left\{\left(1 + 1 - \frac{2\pi}{\pi}\right)\left(\frac{\sin \pi}{n}\right) - \left(\frac{2\pi}{\pi^2} - \frac{2}{\pi}\right)\left(\frac{(-1)^n}{n^2}\right) - 0 + \frac{2}{\pi^2}(0)\right.$$

$$= \left\{\left(\frac{2\pi - 2\pi}{\pi}\right)\left(\frac{\sin \pi}{n}\right) - \left(\frac{2\pi}{\pi^2} - \frac{2}{\pi}\right)\frac{(-1)^n}{n^2} + \frac{2}{\pi^2}(0)\left(1/n^3\right)\right\}$$

$$= -\left(\frac{2\pi}{\pi^2} - \frac{2}{\pi}\right)\left(\frac{(-1)^n}{n^2}\right)$$

$\frac{2}{\pi}$

$\sin n\pi = 0$
$\sin 0 = 0$

$$F_c[f(x)] = \frac{2}{n^2\pi}, \quad n > 0$$

$n = 0$

$$F_c[f(x)] = \int_0^\pi f(x)\,dx$$

$$F_c[f(x)] = \int_0^\pi (1 - x/\pi)^2\,dx$$

$$= \int_0^\pi \left(1 + \frac{x^2}{\pi^2} - \frac{2x}{\pi}\right) dx$$

$$= \left[x + \frac{x^3}{3\pi^2} - \frac{\cancel{2}x^2}{\cancel{2}\pi}\right]_0^\pi$$

$$= \left[\pi + \frac{\pi^{\cancel{3}}}{3\cancel{\pi^2}} - \frac{2\pi^{\cancel{2}}}{2\cancel{\pi}} - (0 + 0 + 0)\right]$$

$$= \left[\pi + \frac{\pi^{\cancel{2}}}{3\cancel{\pi}} - \pi\right] = \left[\cancel{\pi} + \frac{\pi}{3} - \cancel{\pi}\right]$$

$$F_c[f(x)] = \pi/3.$$

$$F_c[f(x)] = \begin{cases} 2/n^2\pi & n > 0 \\ \pi/3 & n = 0 \end{cases}$$

Under what conditions on f(x) fourier transform of $f''(x)$ exist? Find the same when conditions satisfied?

solution:

The fourier transforms of $f(x)$ exist when $f'(x)$ and $f(x)$ vanishes as $x \to \infty$.

$$F\left[f''(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f'' e^{isx}\, dx$$

$u = e^{isx}$ | $dv = f''(x)\,dx$

$du = e^{isx}(is)\,dx$ | $v = f'(x)$.

$$F\left[f''(x)\right] = \frac{1}{\sqrt{2\pi}} \left[\left(f'(x) e^{isx} \right)_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} f'(x) e^{isx} (isx)\, dx \right]$$

Since $f'(x) \to 0$ as $x \to \alpha$.

$$F\left[f''(x)\right] = \frac{1}{\sqrt{2\pi}} \left[-is \int_{-\alpha}^{\alpha} f'(x)\, e^{isx}\, dx \right]$$

$u = e^{isx}$ | $dv = f'(x)\, ds$

$du = (is)\, e^{isx}$ | $v = f(x)$.

$$F\left[f''(x)\right] = \frac{(-is)}{\sqrt{2\pi}} \left[f(x)\, e^{isx} \right]_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} f(x)(is) e^{isx}\, dx.$$

Since $f'(x) \to 0$ as $x \to \alpha$

$$F\left[f''(x)\right] = \frac{(-is)\,(-is)}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(x) e^{isx}\, dx$$

$$F\left[f''(x)\right] = (is)^2\, F(s)$$

Find fourier transform of $\frac{\sin ax}{x}$.

Hence prove that $\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx = a\pi$.

Soln:

Fourier transform of $f(x)$ is

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin ax}{x} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\cos sx + i \sin sx) dx.$$

Since $\frac{\sin ax}{x} \cos sx$ is even fn and $\frac{\sin ax}{x} \sin sx$ is odd function.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin ax}{x} \cos sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{2} \int_{0}^{\infty} \frac{\sin(ax+sx)}{x} + \frac{\sin(ax-sx)}{x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} \frac{\sin(a+s)x}{x} + \frac{\sin(a-s)x}{x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\pi/2 + \pi/2 \right] \text{ if } s+a>0, \; a-s>0$$

$$= \frac{1}{\sqrt{2\pi}} (\pi) \text{ if } s>-a \quad a>s$$

$$F[f(x)] = \begin{cases} \sqrt{\pi/2} & \text{if } -a<s<a \\ 0 & \text{if otherwise} \end{cases}$$

By using Parseval's Identity :

$$\int_{-\alpha}^{\alpha} |f(x)|^2\, dx = \int_{-\alpha}^{\alpha} |F f(x)|^2\, ds$$

i.e) $$\int_{-\alpha}^{\alpha} \frac{\sin^2 ax}{x^2}\, dx = \int_{-a}^{a} \left(\sqrt{\pi/2}\right)^2 ds$$

$$= \pi/2\, [s]_{-a}^{a}$$

$$= \pi/2\, (a+a) = \pi/2\, (2a)$$

$$= a\pi$$

$$\therefore \int_{-\alpha}^{\alpha} \frac{\sin^2 ax}{x^2}\, dx = a\pi .$$

16. Find $F^{-1}\left[\frac{2\sin 3(s-2\pi)}{\sqrt{2\pi}\,(s-2\pi}\right]$

soln :

First we have to find

$$F^{-1}\left[\frac{2\sin 3s}{\sqrt{2\pi}\, s}\right]$$

$$f(x) = F^{-1}\left[\frac{2\sin 3s}{\sqrt{2\pi}\, s}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\alpha}^{\alpha} F(f(x)) e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\alpha}^{\alpha} \frac{2\sin 3s}{\sqrt{2\pi}\cdot s}\cdot e^{-isx}\, ds$$

$$= \frac{1}{2\pi}\times 2 \int_{-\alpha}^{\alpha} \frac{\sin 3s}{s}(\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{\sin 3s}{s}(\cos sx - i\sin sx)\, ds$$

Since $\frac{\sin 3s}{s}\cos sx$ is even function

$\frac{\sin 3s}{s}\sin sx$ is odd function

$$= \frac{2}{\pi} \int_0^{\alpha} \frac{\sin 3s}{s} \cos sx \, ds$$

$$= \frac{2}{\pi} \times \frac{1}{2} \int_0^{\alpha} \left[\frac{\sin (3+x)s}{s} + \frac{\sin (3-x)s}{s} \right] ds$$

$$\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{1}{\pi} \left[\pi/2 + \pi/2 \right] \text{ when } 3+x > 0 \text{ and } 3-x > 0$$

$$= \frac{1}{\pi} (\pi) = 1 \qquad -3 < x < 3$$

$$f(x) = F^{-1} \left[\frac{2 \sin 3x}{\sqrt{2\pi} \cdot 3} \right] = \begin{cases} 1 & |x| < 3 \\ 0 & |x| > 3 \end{cases}$$

By Shifting Theorem:

$$F \left[e^{-ax} F(x) \right] = F(s-a)$$

$$F^{-1} F \left[e^{iax} f(x) \right] = F^{-1} \left[F(s-a) \right]$$

$$= e^{iax} f(x).$$

i.e) $F^{-1} F [s-a]$

i.e) $F^{-1} \left[\frac{2 \sin 3(s-2\pi)}{\sqrt{2\pi} (s-2\pi)} \right]$

$$= e^{-i2\pi x} \begin{cases} 1 & |x| < 3 \\ 0 & |x| > 3 \end{cases}$$

16. Find $f(x)$, if finit sine transform is $\frac{2\pi (-1)^{p-1}}{p^3}$ $\quad 0 < x < \pi$

Soln:

We know that Inverse finite Fourier sine transform is given by

$$\frac{2}{l} \sum^{\alpha} F_s [f(x)] \sin \frac{n\pi x}{l}$$

Here $l = \pi$ and $n = p$

$$f(x) = 4 \sum_{p=1}^{\alpha} \frac{(-1)^{p-1}}{p^3} \sin px.$$

17.) Find $F_c^{-1} f(P)$ if $f(P) = \frac{\cos 2/3\, P\pi}{(2P+1)^2}$, $0 < x < 1$.

Soln:

W.K.T Inverse Finite Fourier cosine transform is given by

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\alpha} F_c f(x) \frac{\cos n\pi x}{l}.$$

Here $l = 1$, $n = P$, $F_c(0) = 1$.

$$f(x) = \frac{1}{1}(1) + \frac{2}{1} \sum_{n=1}^{\alpha} \frac{\cos 2/3\, P\pi}{(2P+1)^2} \cos p\pi x.$$

$$f(x) = \frac{1}{1} + \frac{2}{1} \sum_{n=1}^{\alpha} \frac{\cos 2/3\, P\pi}{(2P+1)^2} \cos p\pi x.$$

Find Fourier sine and cosine transform.

2) If $\begin{cases} 0 & \text{in } 0 < x < \pi/2 \\ -1 & \text{in } \pi/2 < x < \pi \end{cases}$

Soln:

W.K.T $F_s[f(x)] = \int_0^l f(x) \frac{\sin n\pi x}{l} dx.$

$$F_s[f(x)] = \int_0^{\pi} f(x) \frac{\sin n\pi x}{l} dx$$

$$F_s[f(x)] = \int_0^{\pi/2} 0\, dx + \int_{\pi/2}^{\pi} (-1) \frac{\sin n\pi x}{l} dx$$

$$F_s[f(x)] = \int_{\pi/2}^{\pi} - \frac{\sin n\pi x}{\pi} dx$$

$$F_s[f(x)] = \left[\frac{\frac{\cos n x\pi}{\pi}}{n\pi/\pi} \right]_{\pi/2}^{\pi}$$

then we know that

$$F_c\,[f(x)] = \int_0^l f(x)\,.\,\cos\frac{n\pi x}{l}\,dx$$

$$F_c\,[f(x)] = \int_0^\pi f(x)\,\cos\frac{n\pi x}{\pi}\,dx$$

$$= \int_0^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} -1 \cos$$

$$= \int_{\pi/2}^{\pi} -1\,.\,\cos nx\,dx$$

$u = -1 \quad | \quad v = \int \cos nx\,dx$

$u' = 0 \quad | \quad v_1 = +\frac{\sin nx}{n}$

$$= \left[-1\,.\,+\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \left(-\frac{\sin n\pi}{n} + \frac{\sin n\pi/2}{n}\right] \left\{\text{or} \left(\frac{\sin n\pi/2}{n}\right)\right.$$

$$F_c\,[f(x)] = \left[\frac{\sin n\pi/2}{n}\right] = \sin\pi/2 = 1$$

(3). $f(x) = x^3$. $0 < x < 4$.

Solution :

We know that $F_s\,[f(x)] = \int_0^l f(x)\,.\,\frac{\sin n\pi x}{l}\,dx$

$$F_s\,[f(x)] = \int_0^4 x^3\,.\,\frac{\sin n\pi x}{4}\,dx$$

$u = x^3 \quad | \quad v = \int \frac{\sin n\pi x}{4}\,dx$

$u' = 3x^2 \quad | \quad v_1 = -\dfrac{\cos\frac{n\pi x}{4}}{n\pi/4}$

$u'' = 6x$

$u''' = 6 \quad | \quad v_2 = -\dfrac{\sin\frac{n\pi x}{4}}{n^2\pi^2/16}$

$u^{iv} = 0$

$v_3 = \dfrac{\cos\frac{n\pi x}{4}}{n^3\pi^3}$

Bernoulli's formulae,

$$\int u\,dv = uv_1 - u'v_2 + u''v_3 - u'''v_4$$

$$= \left[\frac{x^3\left(-\frac{\cos n\pi x}{4}\right)}{\frac{n\pi}{4}} - \frac{3x^2\left(-\frac{\sin n\pi x}{4}\right)}{\frac{n^2\pi^2}{16}} + \frac{6x\left(\frac{\cos n\pi x}{4}\right)}{\frac{n^3\pi^3}{64}} - \frac{6\left(\frac{\sin n\pi x}{4}\right)}{\frac{n^4\pi^4}{256}} \right]_0^4$$

$$= \left\{ \frac{-(4^3)\frac{\cos 4\pi n}{4}}{\frac{n\pi}{4}} + \frac{3(4^2)\frac{\sin 4\pi n}{4}}{\frac{n^2\pi^2}{16}} + \frac{6(4)\frac{\cos 4n\pi}{4}}{\frac{n^3\pi^3}{64}} - \frac{6\frac{\sin 4\pi n}{4}}{\frac{n^4\pi^4}{256}} \right\}$$

$$= \left\{ -256\frac{\cos n\pi}{n\pi} + 0 + 1536\frac{\cos n\pi}{n^3\pi^3} \right\}$$

$$= \left\{ -256\frac{(-1)^n}{n\pi} + 1536\frac{(-1)^n}{n^3\pi^3} \right\}$$

$$= 256\left\{ \frac{6(-1)^n}{n^3\pi^3} - \frac{(-1)^n}{n\pi} \right\}$$

$$F_s[f(x)] = \frac{256}{n\pi}(-1)^n\left[\frac{6}{n^2\pi^2} - 1\right]$$

Then we know that.

$$F_c[f(x)] = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$F_c[f(x)] = \int_0^4 x^3 \cos\frac{n\pi x}{4}\, dx$$

$u = x^3$ | $v = \int \cos\frac{n\pi x}{4}\, dx$

$u' = 3x^2$ | $v_1 = \dfrac{\sin\frac{n\pi x}{4}}{n\pi/4}$

$u'' = 6x$ | $v_2 = \dfrac{-\cos\frac{n\pi x}{4}}{n^2\pi^2/16}$

$u''' = 6$ | $v_3 = \dfrac{-\sin\frac{n\pi x}{4}}{n^3\pi^3/64}$

$u^{iv} = 0$ | $v_4 = \dfrac{\cos\frac{n\pi x}{4}}{n^4\pi^4/256}$

$$= \left[\frac{x^3\left(\sin\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}} - \frac{(3x^2)\left(-\cos\frac{n\pi x}{4}\right)}{n^2\pi^2/16} + \frac{6x\left(-\cancel{\cos}\sin\frac{n\pi x}{4}\right)}{\frac{n^3\pi^3}{64}} - \frac{6\left(\cos\frac{n\pi x}{4}\right)}{\frac{n^4\pi^4}{256}} \right]_0^4$$

$$= \left\{ \frac{4^3\left(\sin\frac{4n\pi}{4}\right)}{n\pi/4} + \frac{3(4^2)\cos\frac{4n\pi}{4}}{n^2\pi^2/16} + \frac{-6(4)\sin 4n\pi}{\frac{n^3\pi^3}{64}} - \frac{6\cos 4n\pi/4}{\frac{n^4\pi^4}{256}} + 0 + 0 + 0 + \frac{6\,\frac{\cos 0}{4}}{\frac{n^4\pi^4}{256}} \right\}$$

$$\begin{array}{r} 16 \\ 3 \\ \hline 48 \end{array} \qquad \begin{array}{r} 48 \\ \times 16 \\ \hline 288 \\ 48 \\ \hline 768 \end{array}$$

$$= \left\{ 0 + \frac{768\cos n\pi}{n^2\pi^2} + 0 - 1536\,\frac{\cos n\pi}{n^4\pi^4} \right.$$

$$= 3 \cdot 4^4 \left\{ \frac{(-1)^n}{n^2\pi^2} - \frac{2(-1)^n}{n^4\pi^4} + \frac{2}{n^4\pi^4} \right\}$$

$$= \frac{3 \cdot 4^4}{n^2\pi^2} \left[(-1)^n \left(1 - \frac{2}{n^2\pi^2}\right) + \frac{2}{n^4\pi^4} \right]$$

$$F_c\left[f(x)\right] = \frac{3 \cdot 4^4}{n^2\pi^2} \left[(-1)^n + \frac{2}{n^2\pi^2}\left(1-(-1)^n\right) \right]$$

4. Find $F_c\left[f(x)\right]$ if $f(x) = \sin ax$ in $0 < x < \pi$

Soln:

$$F_c\left[f(x)\right] = \int_0^l f(x) \frac{\cos n\pi x}{l}\, dx$$

$$= \int_0^\pi \sin ax \cdot \cos \frac{n \cancel{\pi} x}{\cancel{l}}\, dx$$

$$= \int_0^\pi \sin ax \cdot \cos nx\, dx$$

$$\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{1}{2} \int_0^\pi \sin(a+n)x + \sin(ax-nx)\, dx$$

$$= \frac{1}{2} \int_0^\pi \sin(a+n)x + \sin(a-n)x\, dx$$

$$= \frac{1}{2} \left[\frac{-\cos(a+n)x}{a+n} - \frac{\cos(a-n)x}{a-n} \right]_0^\pi$$

$$= \frac{1}{2} \left[\frac{-\cos(a+n)\pi}{a+n} - \frac{\cos(a-n)\pi}{a-n} + \frac{1}{a+n} + \frac{1}{a-n} \right]$$

$$= -\frac{1}{2}\left[\left(\frac{\cos(a+n)\pi}{a+n} + \frac{\cos(a-n)\pi}{a-n}\right) + \left(\frac{1}{a+n} + \frac{1}{a-n}\right)\right]$$

$$= -\frac{1}{2}\left\{\left[\frac{\cos(a+n)\pi}{a+\pi} + \frac{\cos(a-n)\pi}{a-n}\right] - \left[\frac{a-\cancel{n}+a+\cancel{n}}{a^2 n^2}\right]\right\}$$

case - I

If 'a' is not an integer

$$F_c\,[f(x)] = -\frac{1}{2}\left\{\left[\frac{\cos a\pi \cdot \cos n\pi - \sin a\pi \cdot \sin n\pi}{a+n}\right] + \left[\frac{\cos a\pi \cos n\pi + \sin a\pi \cdot \sin n\pi}{a-n}\right] - \frac{2a}{a^2-n^2}\right\}$$

$$= -\frac{1}{2}\left[\frac{\cos a\pi\,(-1)^n}{a+n} + \frac{\cos a\pi\,(-1)^n}{a-n} - \frac{2a}{a^2-n^2}\right]$$

$$= -\frac{1}{2}\cos a\pi\,(-1)^n\left(\frac{1}{a+n} + \frac{1}{a-n}\right) + \frac{a}{a^2-n^2}$$

$$= -\frac{1}{2}\cos a\pi\,(-1)^n\left(\frac{a-\cancel{n}+a+\cancel{n}}{a^2-n^2}\right) + \frac{a}{a^2-n^2}$$

$$= -1\,\frac{\cos a\pi}{2}\,(-1)^n\left(\frac{2a}{a^2-n^2}\right) + \frac{a}{a^2-n^2}$$

$$F_c\left[\sin ax\right] = \frac{2a}{a^2-n^2}$$

$$\therefore F_c\left[\sin ax\right] = \begin{cases} 0 & \text{if } n, a \text{ is odd / even} \\ 2a/(a^2-n^2) & \text{if } n \text{ is odd, } a \text{ is even} \end{cases}$$